Lecture 03

Rotation and Translation

Acknowledgement :
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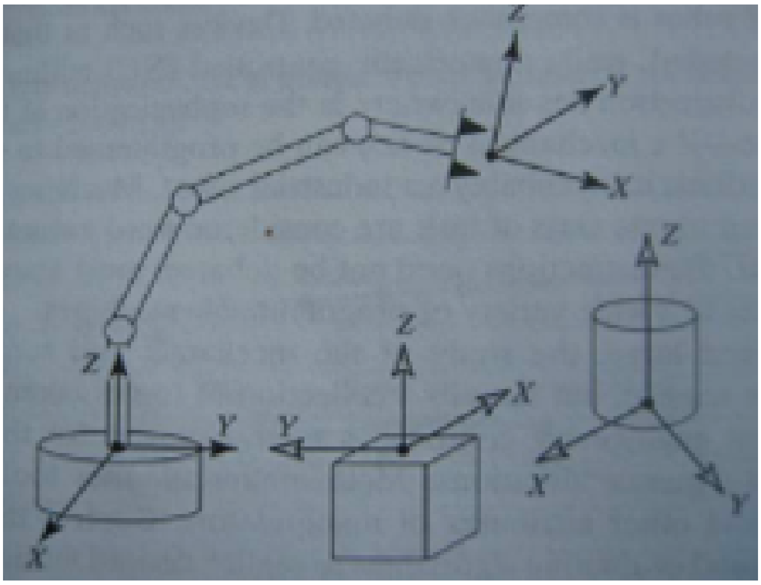
Spatial Descriptions

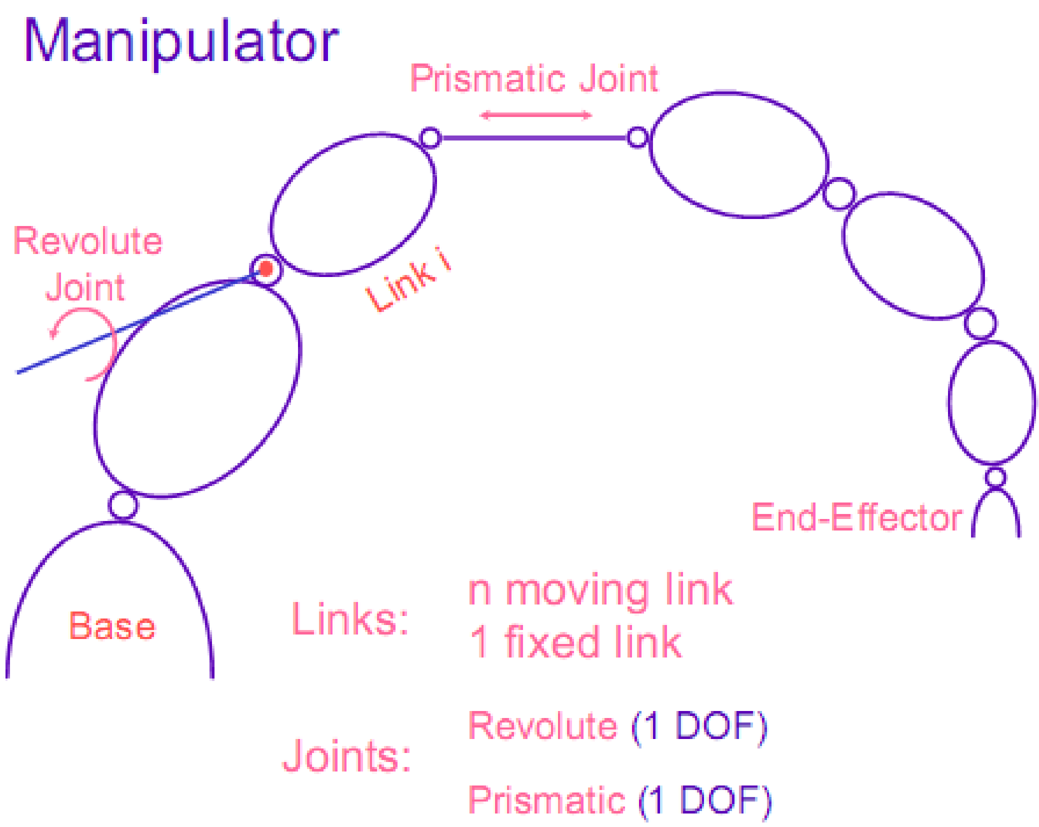
- Manipulator position and orientation should be described mathematically.
- All the parts of the manipulator, including the end-effector and links, and the other surrounding objects it deals with are also to be described using mathematical relationships.
- Joint actuators are controlled according to the spacial position and orientation of the manipulator

Body Frames

- To describe position and orientation of an object, a 3D co-ordinate frame is attached to that object (Body frame, link frame, tool frame..). Then, the position of the object is the position of the origin of this frame, and orientation is the directions of its axes, w.r.t a world co-ordinate frame.

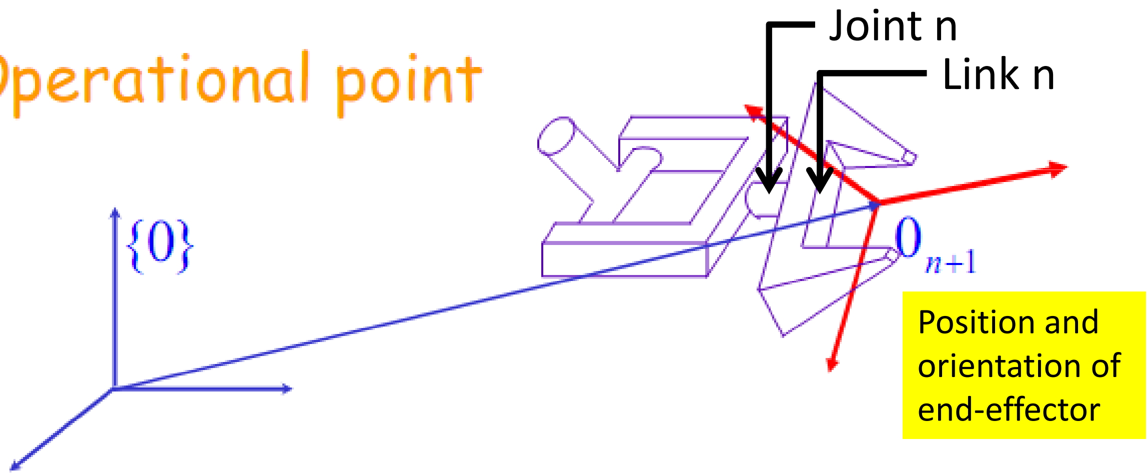


Spatial Descriptions



Operational Co-ordinates

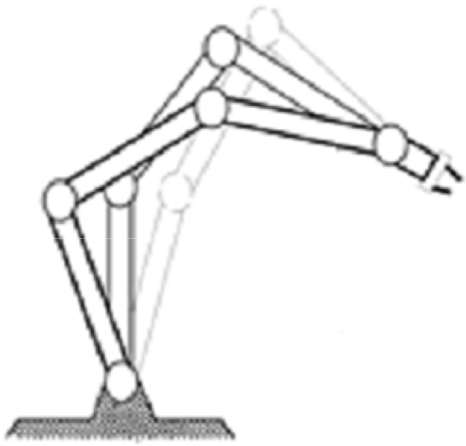
O_{n+1} : Operational point



m : Number of independent variables required to describe the position and orientation of the end-effector

In 3D space m=6 (3 for x,y,z position, and 3 more for direction control)

Arm Redundancy



A robot arm is said to be redundant if $n>m$

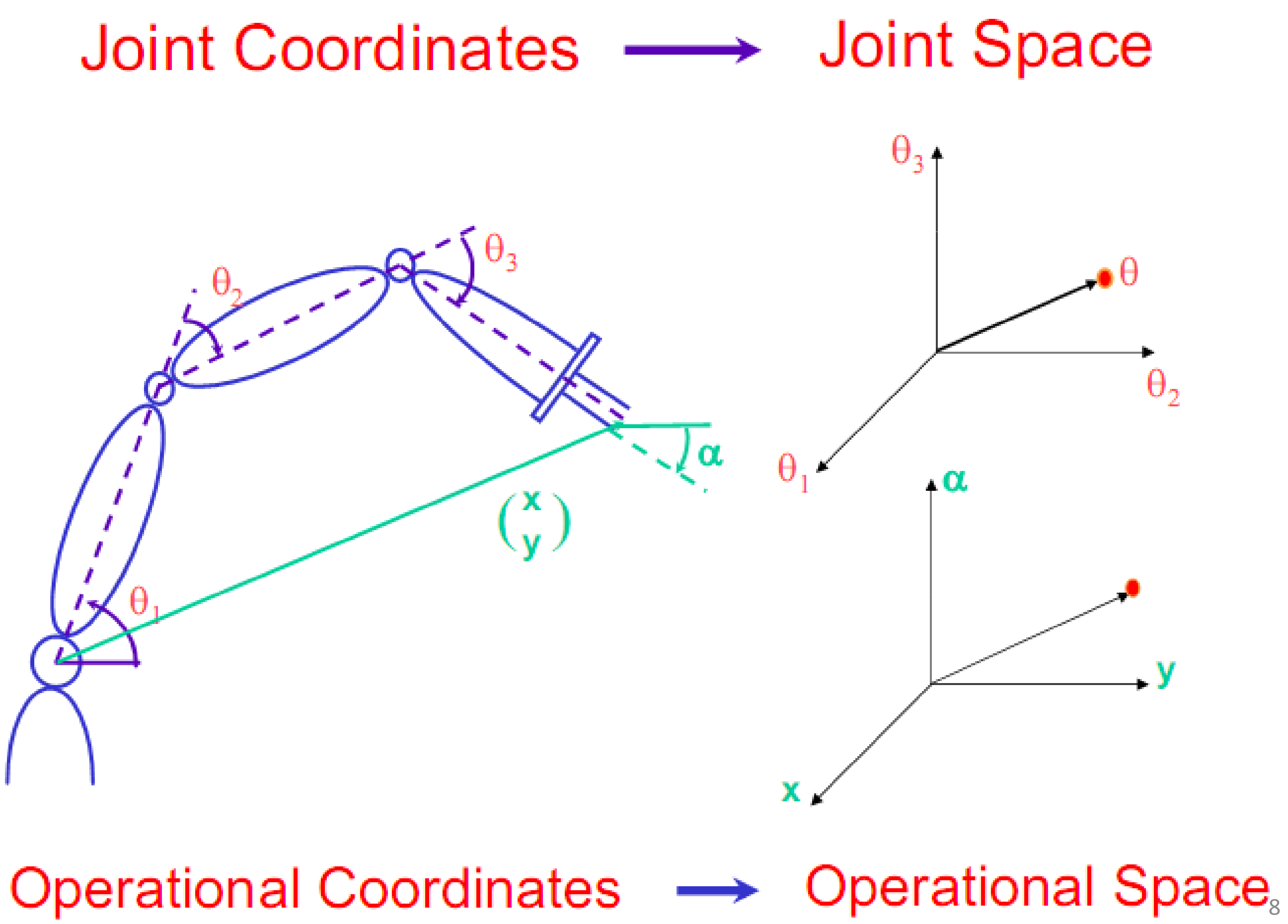
Degrees of redundancy = n-m

- A manipulator arm must have at least 6 degrees of freedom in order to locate its end-effecter at an arbitrary position with an arbitrary orientation.
- If a manipulator has more than 6 degrees of freedom, there exist an infinite number of solutions to the kinematic equation.

Arm Redundancy

- The human arm, which has 7 degrees of freedom from shoulder to wrist. Place the palm firmly on a plane surface and move the elbow joint continuously without moving the shoulder joint. It implies the existence of an infinite number of arm configurations (solutions to the inverse kinematic problem) of the human arm for a given hand position/orientation.
- Manipulator arms with more than six degrees of freedom are referred to as ***redundant manipulators***

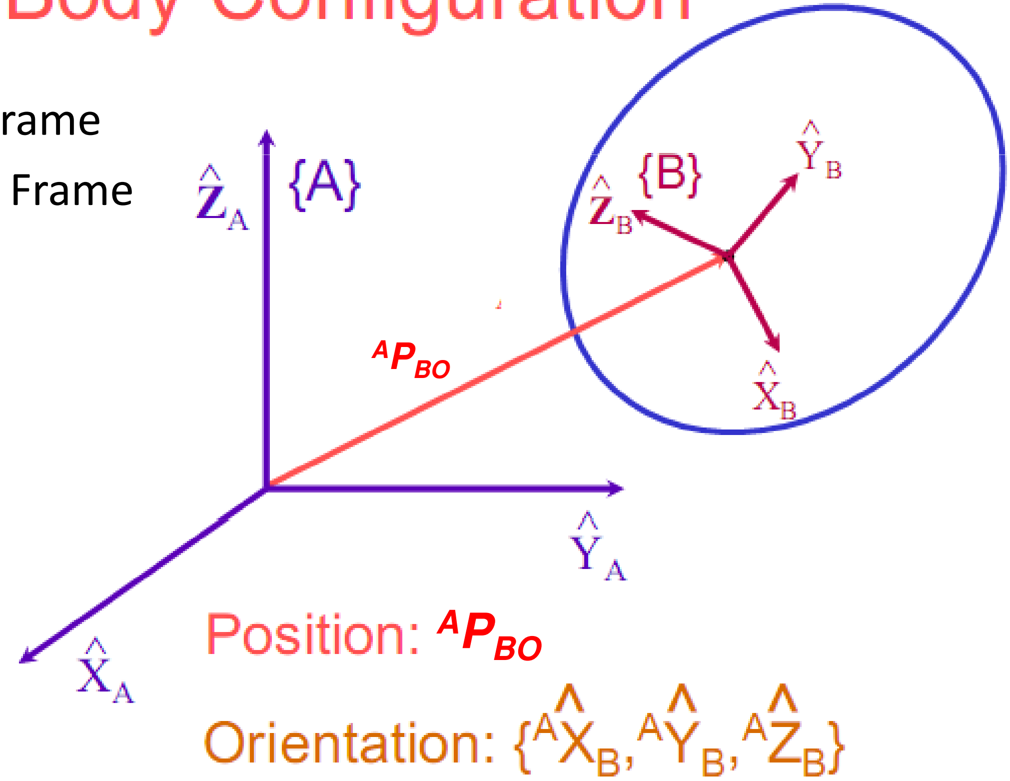
Joint and Operational Spaces



Co-ordinate Frames for Links

Rigid Body Configuration

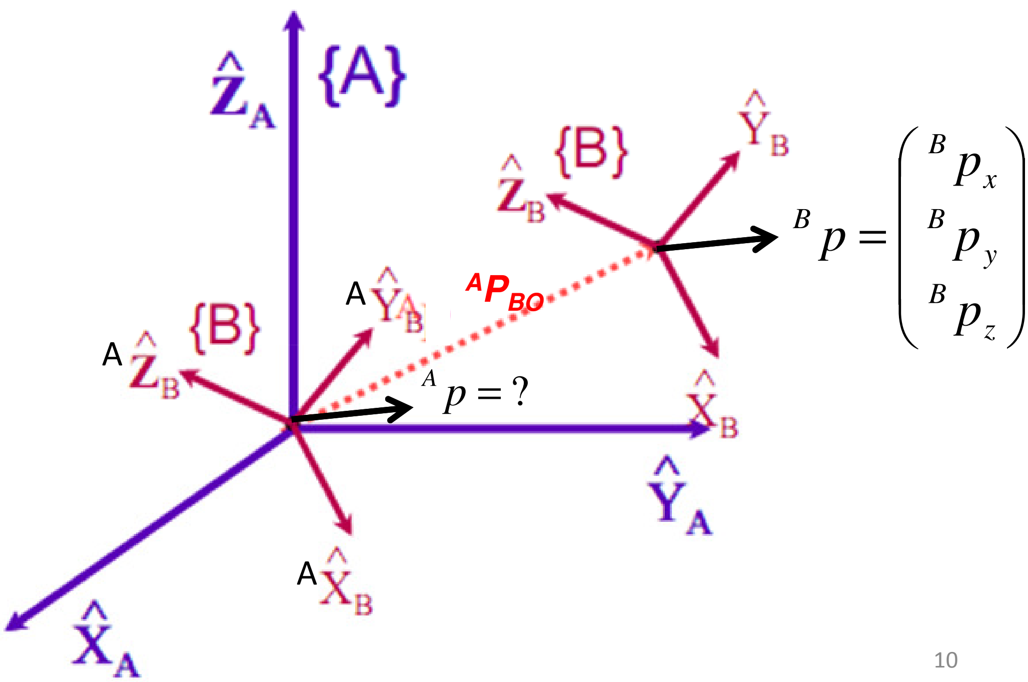
- {B} : Relative Frame
- {A} : Reference Frame



Description of {B} w.r.t {A}

Orientation of {B} w.r.t {A}

- Imagine a vector $^B P=(^B p_x, ^B p_y, ^B p_z)^T$ described in {B}
- Translate it to frame {A} origin, and
- Disregard translation $^A P_{BO}$ and determine the vector w.r.t {A}



Derivation of Rotation Matrix

$$^{A}p_x = {}^{B}p_x \hat{x}_B.\hat{x}_A + {}^{B}p_y \hat{y}_B.\hat{x}_A + {}^{B}p_z \hat{z}_B.\hat{x}_A$$

$$^{A}p_y = {}^{B}p_x \hat{x}_B.\hat{y}_A + {}^{B}p_y \hat{y}_B.\hat{y}_A + {}^{B}p_z \hat{z}_B.\hat{y}_A$$

$$^{A}p_z = {}^{B}p_x \hat{x}_B.\hat{z}_A + {}^{B}p_y \hat{y}_B.\hat{z}_A + {}^{B}p_z \hat{z}_B.\hat{z}_A$$

$$\begin{pmatrix} {}^{A}p_x \\ {}^{A}p_y \\ {}^{A}p_z \end{pmatrix} = \begin{pmatrix} \hat{x}_B.\hat{x}_A & \hat{y}_B.\hat{x}_A & \hat{z}_B.\hat{x}_A \\ \hat{x}_B.\hat{y}_A & \hat{y}_B.\hat{y}_A & \hat{z}_B.\hat{y}_A \\ \hat{x}_B.\hat{z}_A & \hat{y}_B.\hat{z}_A & \hat{z}_B.\hat{z}_A \end{pmatrix} \begin{pmatrix} {}^{B}p_x \\ {}^{B}p_x \\ {}^{B}p_z \end{pmatrix}$$

$$^{A}p = \begin{pmatrix} {}^{A}\hat{x}_B & {}^{A}\hat{y}_B & {}^{A}\hat{z}_B \end{pmatrix} {}^{B}p$$

$$^{A}p = {}^{A}_{B}R \, {}^{B}p$$

Derivation of Rotation Matrix

$${}^{A}_{B}R = \begin{pmatrix} {}^{A}\hat{x}_B & {}^{A}\hat{y}_B & {}^{A}\hat{z}_B \end{pmatrix}$$

$${}^{A}_{B}R = \begin{pmatrix} \hat{x}_B.\hat{x}_A & \hat{y}_B.\hat{x}_A & \hat{z}_B.\hat{x}_A \\ \hat{x}_B.\hat{y}_A & \hat{y}_B.\hat{y}_A & \hat{z}_B.\hat{y}_A \\ \hat{x}_B.\hat{z}_A & \hat{y}_B.\hat{z}_A & \hat{z}_B.\hat{z}_A \end{pmatrix} = \begin{pmatrix} {}^{B}\hat{x}_A^T \\ {}^{B}\hat{y}_A^T \\ {}^{B}\hat{z}_A^T \end{pmatrix} = {}^{B}_{A}R^T$$

$${}^{A}_{B}R = {}^{B}_{A}R^T$$

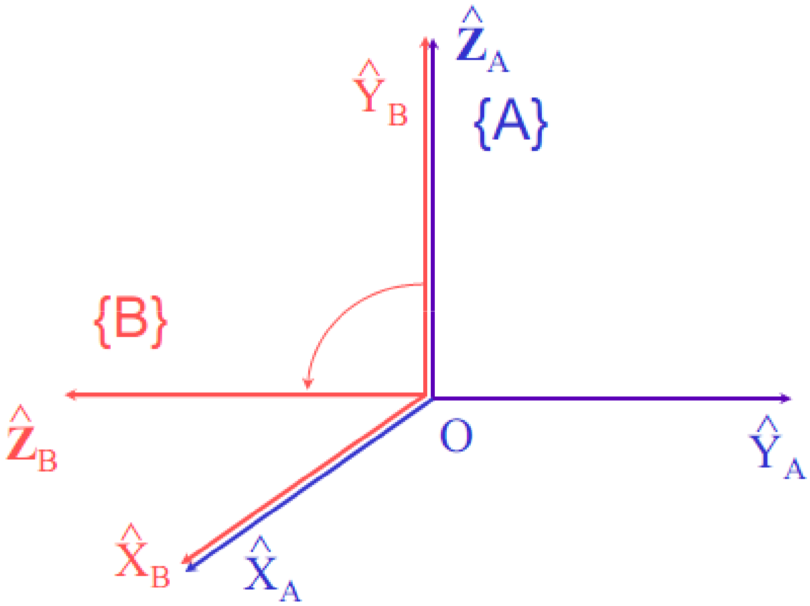
Inverse of Rotation Matrices

$${}^{A}_{B}R^{-1} = {}^{B}_{A}R = {}^{A}_{B}R^T$$

$${}^{A}_{B}R^{-1} = {}^{A}_{B}R^T$$

Orthonormal Matrix

Example

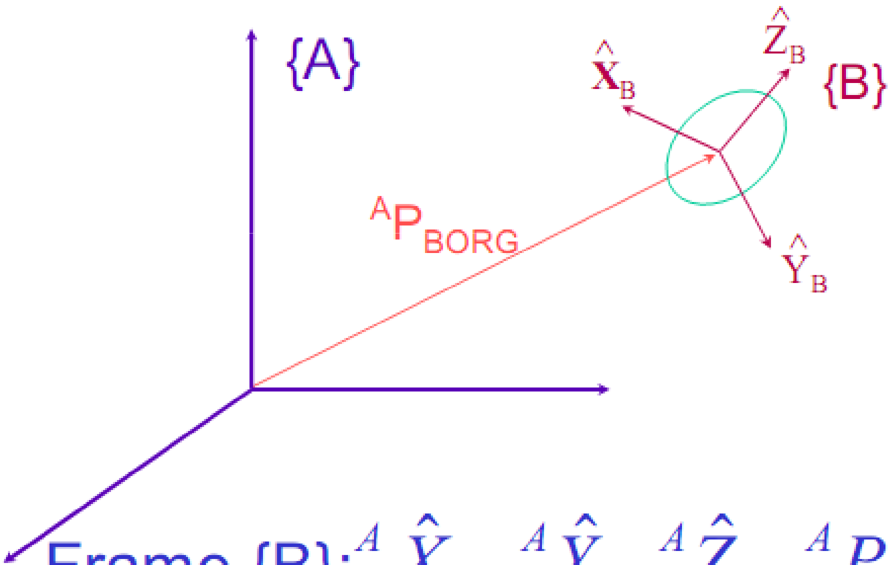


$$
{}^{A}_{B}R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}
\begin{matrix} \leftarrow {}^{B}\hat{X}_A^T \\ \leftarrow {}^{B}\hat{Y}_A^T \\ \leftarrow {}^{B}\hat{Z}_A^T \end{matrix}
$$

$\uparrow \quad \uparrow \quad \uparrow$

${}^{A}\hat{X}_B \quad {}^{A}\hat{Y}_B \quad {}^{A}\hat{Z}_B$

Frame Description

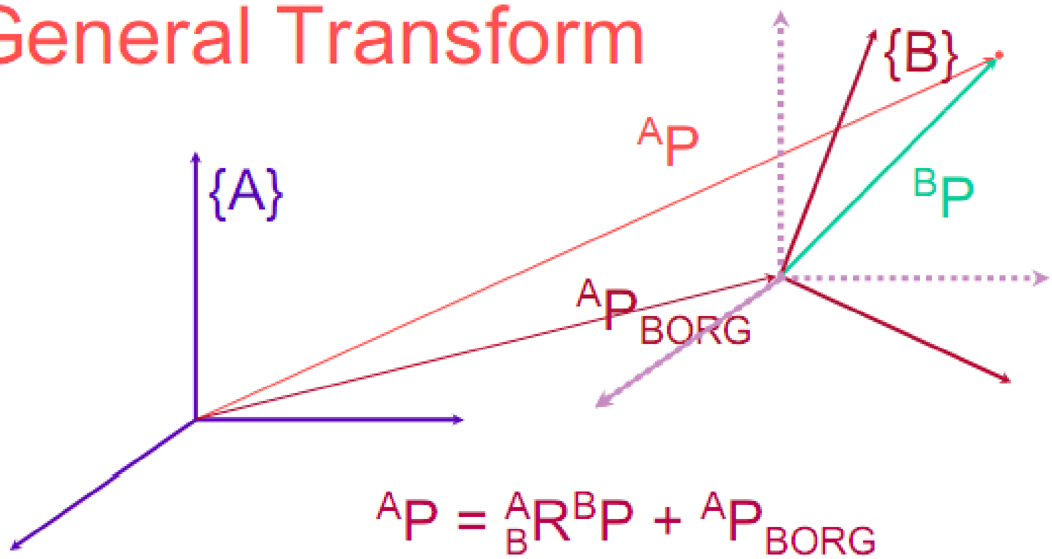


Frame {B}: ${}^{A}\hat{X}_B, {}^{A}\hat{Y}_B, {}^{A}\hat{Z}_B, {}^{A}P_{Borg}$

$$\{B\} = \{ {}^{A}_{B}R \quad {}^{A}P_{Borg} \}$$

Frame Description as a Matrix Transformation

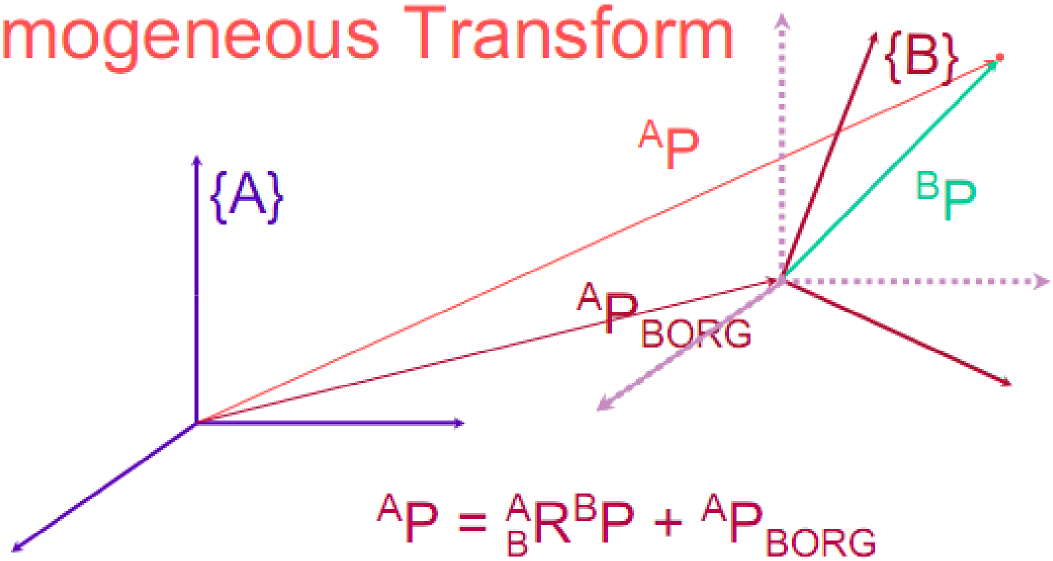
General Transform



$${}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG}$$

Can the rotation and translation be represented by one matrix transformation?

Homogeneous Transform



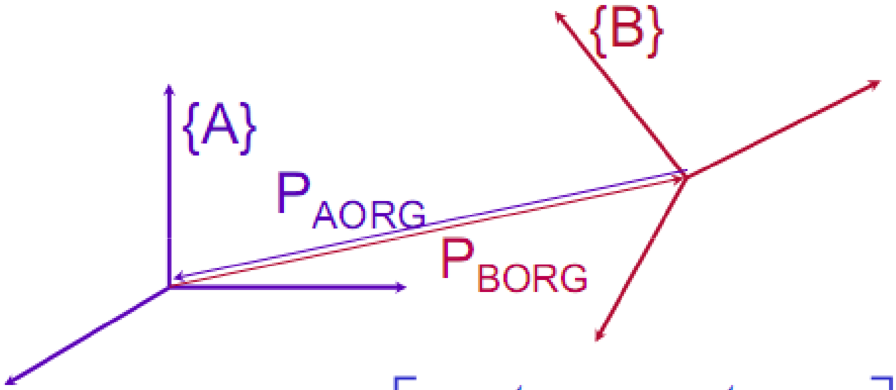
$${}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG}$$

$$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{Borg} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}$$

Homogeneous Transformation Matrix

$$\underset{(4\times1)}{{}^{A}P} = \underset{(4\times4)}{{}^{A}_{B}T}\ \underset{(4\times1)}{{}^{B}P}$$

Inverse Transform

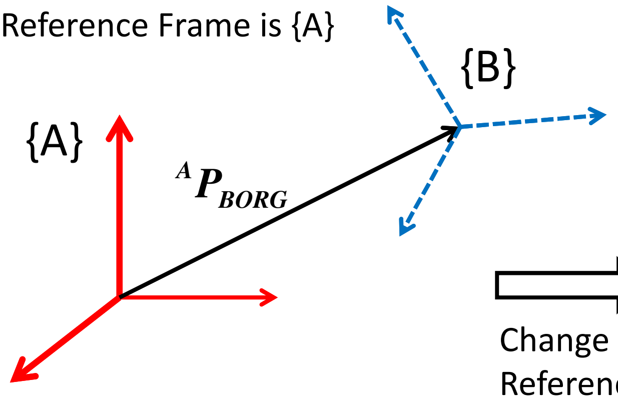


$$ {}^{A}_{B}T = \begin{bmatrix} & {}^{A}_{B}R & & {}^{A}P_{Borg} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

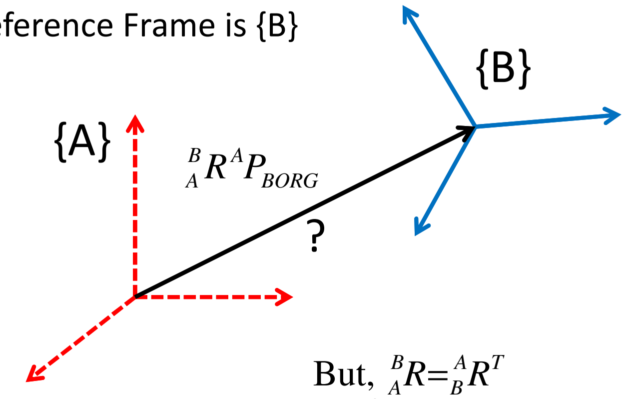
Can ${}^{B}_{A}T = {}^{A}_{B}T^{-1}$ be determined from ${}^{A}_{B}T$?

Inverse Homogeneous Transformation

Reference Frame is {A}

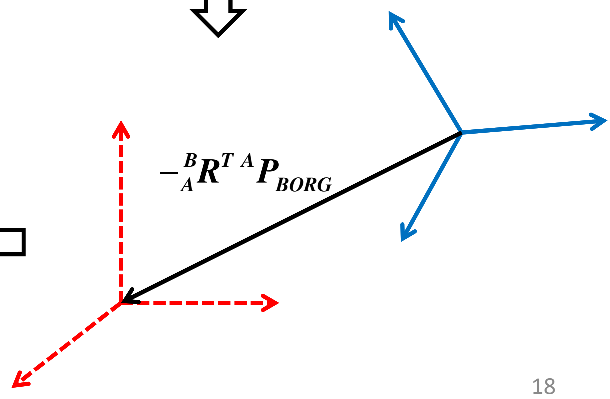


Reference Frame is {B}



Change Reference Frame

But, ${}^{B}_{A}R = {}^{A}_{B}R^{T}$
And, vector direction has to be reversed

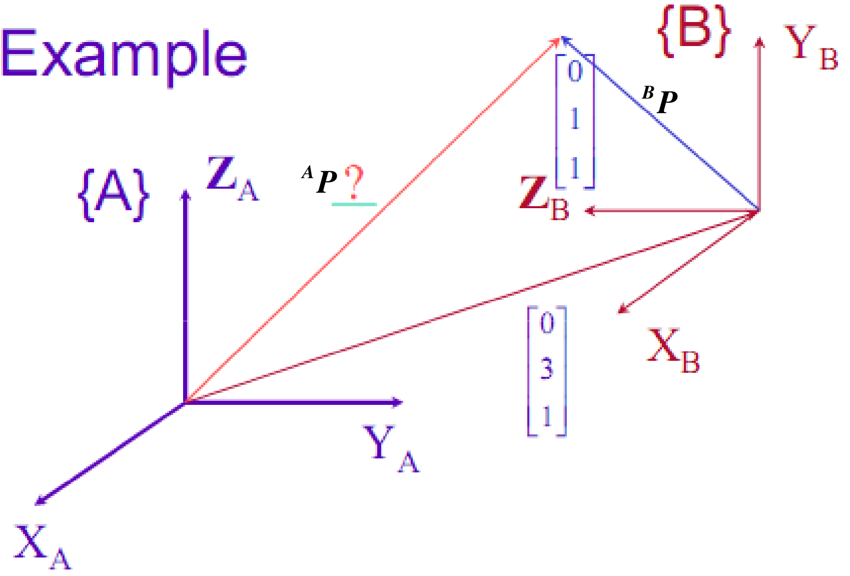


Inverse Homogeneous Transformation Matrix

$$ {}^{B}_{A}T = \begin{pmatrix} & {}^{B}_{A}R & & {}^{B}P_{AORG} \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{B}_{A}T = \begin{pmatrix} & {}^{A}_{B}R^{T} & & -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

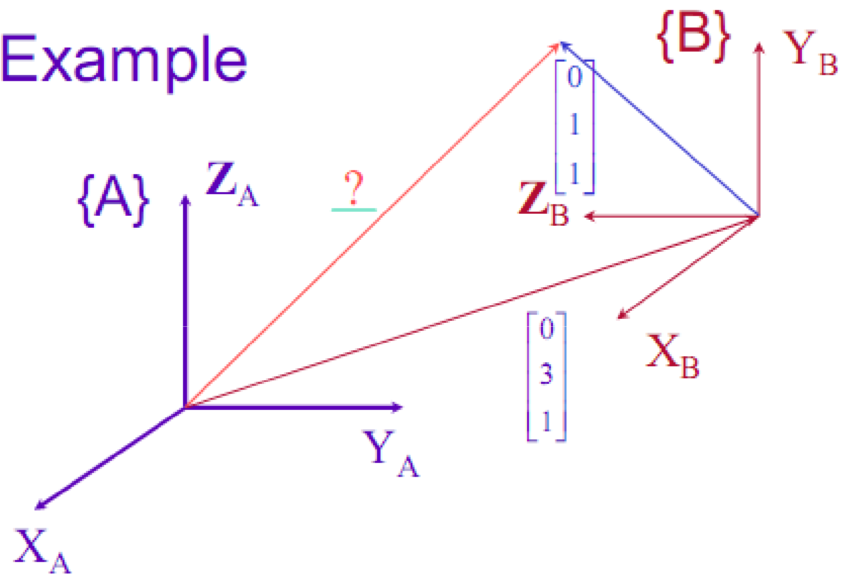
Example



Determine the Homogeneous Transformation Matrix ${}^{A}_{B}T$ and vector ${}^{A}P$

Determine the Inverse Homogeneous Transformation Matrix ${}^{B}_{A}T$

Example



Homogeneous Transform

$$ {}^{A}_{B}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{B}P = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} $$

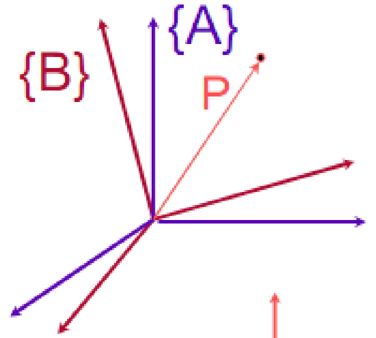
$$ {}^{A}P = {}^{A}_{B}T \cdot {}^{B}P \Rightarrow {}^{A}P = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix} $$

Operators

Mapping: changing descriptions from frame to frame

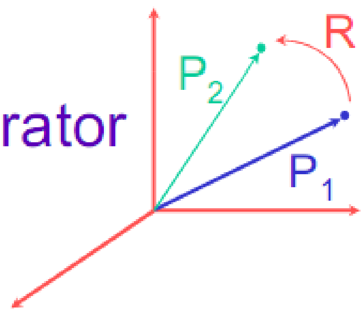
Operators: moving points (within the same frame)

Mapping



$$^{A}P = {}^{A}_{B}R \; {}^{B}P$$

Rotational Operator



$$R: \; P_1 \longrightarrow P_2$$

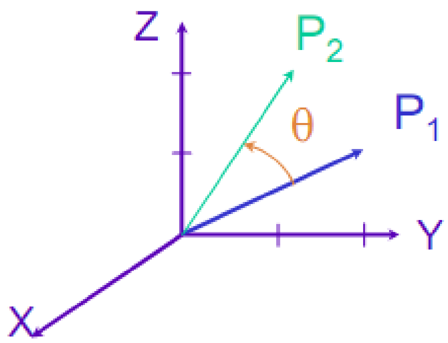
$$\boxed{P_2 = R \; P_1}$$

Rotational Operators

$$R_K(\theta): \quad P_1 \longrightarrow P_2$$

$$P_2 = R_K(\theta) \, P_1$$

Example

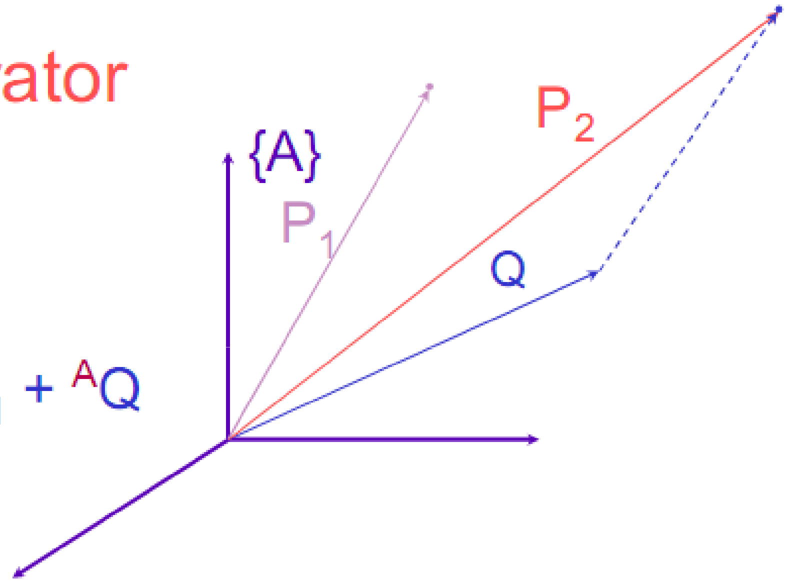


$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$\theta_x = 30°$

$$P_2 = R_X(\theta) P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8 & -0.6 \\ 0 & 0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Translation Operator

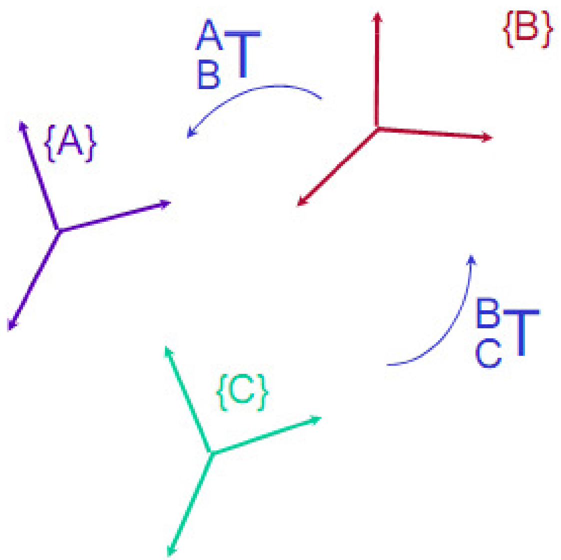


Operator: ${}^{A}P_2 = {}^{A}P_1 + {}^{A}Q$

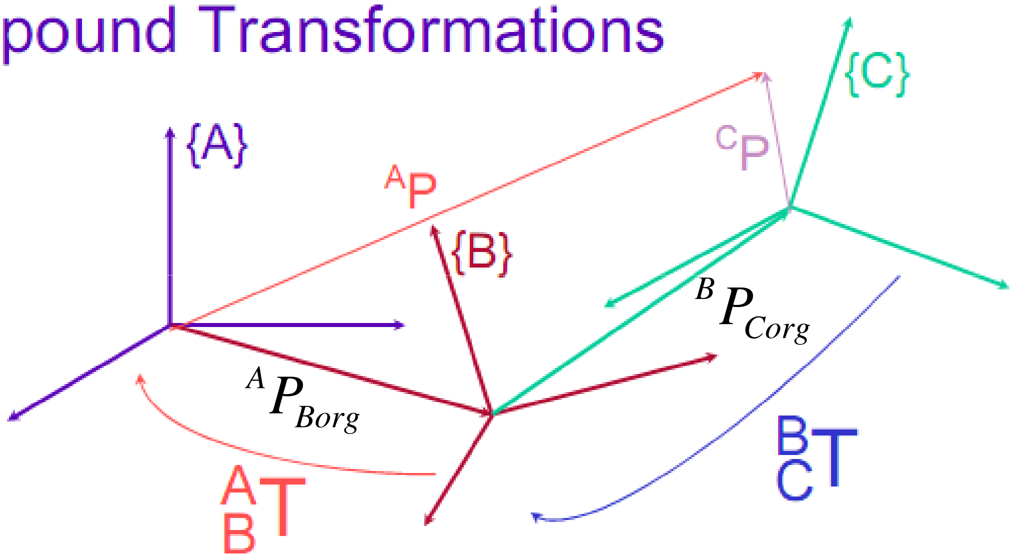
Homogeneous Transform:

$$D_Q = \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow {}^{A}P_2 = {}^{A}D_Q\,{}^{A}P_1$$

Transform Equation



Compound Transformations



$$^{B}P = {}^{B}_{C}T\, {}^{C}P \quad --(2)$$

$$^{A}P = {}^{A}_{B}T\, {}^{B}P \quad --(1)$$

$$^{A}P = {}^{A}_{B}T\, {}^{B}_{C}T\, {}^{C}P \Rightarrow {}^{A}_{C}T = {}^{A}_{B}T\, {}^{B}_{C}T$$

Compound Homogeneous Transformation Matrix

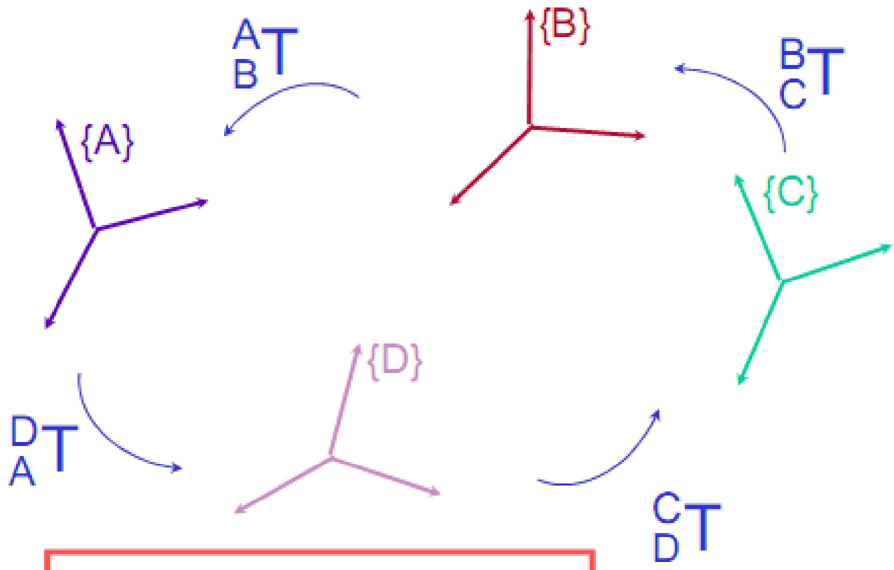
$$^{A}_{C}T = {}^{A}_{B}T\, {}^{B}_{C}T$$

$$^{A}_{C}R = {}^{A}_{B}R\, {}^{B}_{C}R \qquad {}^{A}P_{Corg} = ?$$

$$= {}^{A}P_{Borg} + {}^{A}_{B}R\, {}^{B}P_{Corg}$$

$$^{A}_{C}T = \begin{bmatrix} {}^{A}_{B}R\, {}^{B}_{C}R & {}^{A}_{B}R\, {}^{B}P_{Corg} + {}^{A}P_{Borg} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

Transform Equation



$$^{A}_{B}T\,^{B}_{C}T\,^{C}_{D}T\,^{D}_{A}T = I$$

$$\Rightarrow \quad ^{B}_{A}T = {}^{B}_{C}T\,^{C}_{D}T\,^{D}_{A}T$$